2024 AGMC Tianyi Summer Cup Math Competition

【Individual Round - Senior Group】

Question Sheet

(Full Marks: 100 points Duration: 4 hours)

Read the following instructions carefully before you start the exam.

I. The competition consists of 14 questions with a total score of 100 points and a duration of 4 hours. It includes two sessions: the first session from 10:00 AM to 11:20 AM for Multiple-choice and Fill-in-the-blank questions; and the second session from 2:00 PM to 4:40 PM for Free-response questions.

II. Answers should be written on the answer sheet. Answers written on the question sheet or draft paper will be considered invalid.

III. This competition is an online open-book exam, and consulting paper materials is allowed. The use of electronic devices such as computers, mobile phones, calculators, etc. is prohibited. Any use of electronic devices will result in disqualification.

IV. This competition is an individual round. Consequently, any collaboration is prohibited. Any forms of collaboration will result in disqualification.

V. After the time is up, please take a photo of the answer sheet and upload it to Rainclassroom. Late submissions will be considered invalid.

Section A【Multiple Choice Questions】

(*This section contains 5 questions, each worth 4 points, for a total of 20 points*)

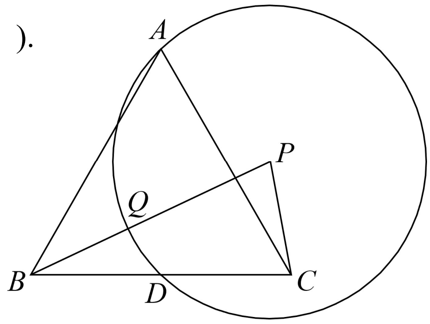
1. The number of real roots of the equation $x^2 = 2024\sin(\pi x)$ is ( ).

A. 44 B. 45 C. 88 D. 90

2. $f(x)$ is a function which is monotonically increasing, $f: \mathbb{N}^+ \to \mathbb{N}^+$, $f(f(x)) = 2x + 1$, then we have $f(2024) =$ ( ).

A. 3025 B. 3026 C. 3036 D. 3037

3. Let $\triangle ABC$ be an equilateral triangle with side length 4. Point D is the midpoint of BC, and point P is a point on the right side of line AC. $\odot P$ passes through points A and D. Connect BP and CP, and let the $\odot P$ intersect BP at point Q. The maximum distance from point Q to line CP is ( ).



A. 2 B. $\sqrt{5}$

C. $\sqrt{6}$ D. $\sqrt{7}$

4. Three positive roots of the equation $ax^3 + bx^2 + cx + d = 0$ are x_1, x_2, and x_3, respectively. $a + b = c + d$. Let $f(x) = -\frac{4}{\sqrt{x^2+1}}$, then the maximum value of $\sqrt{3\sqrt{2}f(x_1) + 17} + \sqrt{\sqrt{15}f(x_2) + \frac{31}{2}} + \sqrt{\sqrt{15}f(x_3) + \frac{31}{2}}$ is ( ).

A. 2 B. $\sqrt{5}$ C. $\sqrt{6}$ D. $\sqrt{7}$

5. Definition: For a ten-digit number, if $0 \sim 9$ each appear exactly once in its each digits, we call such a number a *Beez number***.** A *Beez pair* consists of two *Beez numbers* where one *Beez number* is exactly twice the other *Beez number*. The number of *Beez pairs* is ( ).

A. 114514 B. 122880 C. 147456 D. 184320

Section B【Fill in the Blanks】

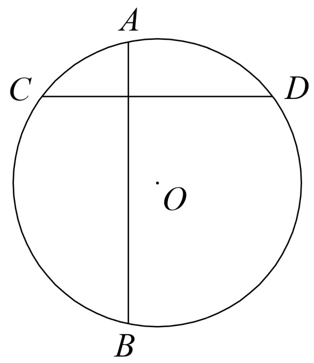
(*This section contains 5 questions, each worth 4 points, for a total of 20 points*)

6. Point A is on the x-axis. Lines AB and BC are tangent to the graph of the function $y = x^3$ at points B and C, respectively. $AB = AC$, $x_A =$ __________ .

7. Let p be a three-digit prime number. The sum of the digits of p and p^2 are equal. Compute $p_{\min} =$ __________ .

8. Given that $m, n \in \mathbb{N}^+$, and each a_i is independently selected as a positive integer less than m, the probability of satisfying $m \mid \sum_{i=1}^{n} a_i$ is __________ . (use explicit expression in terms of m, n)

9. Let $\odot O$ be a circle with a radius of 5, AB and CD are two chords that are mutually perpendicular in $\odot O$, where $AB = 4\sqrt{6}$, $CD = 8$. Within $\odot O$, there exists a point P such that the circumcircle of $\triangle ACP$ is tangent to the circumcircle of $\triangle BDP$, and the circumcircle of $\triangle BCP$ is tangent to the circumcircle of $\triangle ADP$. $OP =$__________ .



10. Prushka independently and randomly select n points on the circumference of a circle, then connect each pair of adjacent points to construct an n-sided polygon. The probability that all interior angles of this polygon are obtuse is __________ . (use explicit expression in terms of n)

Section C【Free Response Questions】

(*This section contains 4 questions, each worth 15 points, for a total of 60 points*)

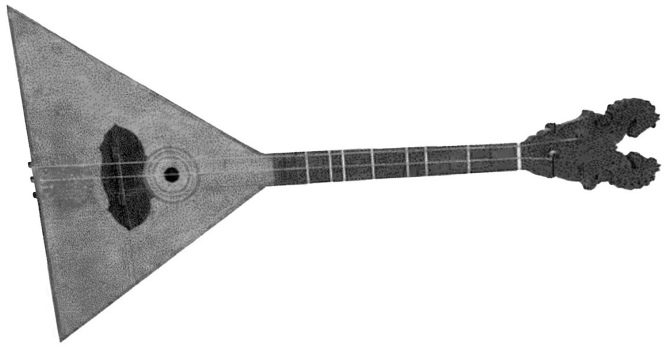
Geometry Part

11. Furina not only loves mathematics, but is also interested in music, therefore, she plans to learn to play a traditional musical instrument.

Orchestration's Prelude

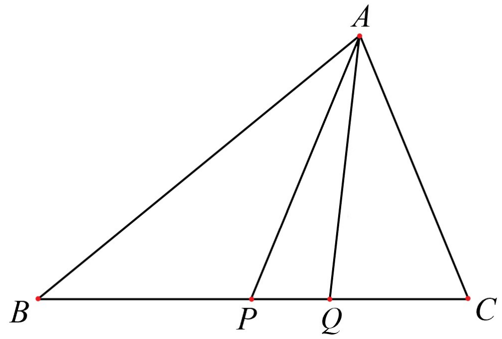
In musical composition, *Retrograde* and *Inversion* are common techniques used for transforming melodies: *Retrograde* refers to playing a melody backward, creating a left-to-right symmetrical mirror effect on the melody; *Inversion* refers to flipping a melody upside down, creating an up-and-down symmetrical mirror effect. The application of these two techniques on a music composition epitomizes the concept of two *Isogonal Lines* in geometry world.

Given an acute triangle $\triangle ABC$, if there exist points P and Q on side BC such that $\angle BAP = \angle CAQ$, then AP and AQ are called the *Isogonal Lines* of $\triangle ABC$. (we will only discuss *Isogonal Lines* in acute triangles for convenience)



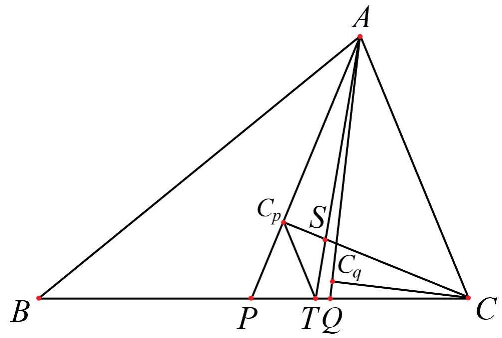
I am in need of music that would flow

[**2**] *балалайка* is a type of string instrument from Russia with a triangular body and three strings. Assuming that the strings have equal amplitudes when they vibrate, which means the two lines are *Isogonal Lines*. We abstract this situation into the geometric model mentioned before. Proof: $\frac{AB^2}{AC^2}=\frac{BP\cdot BQ}{CP\cdot CQ}$.



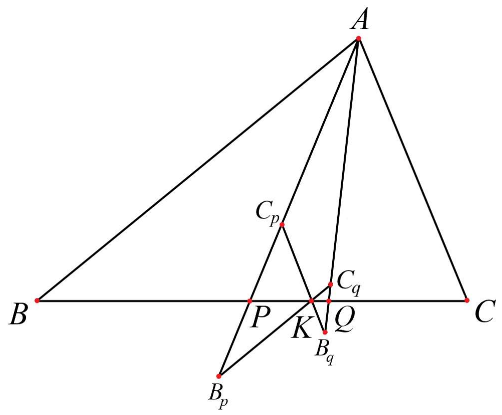
Over my fretful, feeling fingertips,

[**3**] The optimal angle for plucking the string is when it is perpendicular. There is a method to find a sound wave circle: C_p, C_q are projection of point C on AP, AQ, respectively. Construct $TC_p \,//\, AC$ and let it intersect BC at point T. Line AT intersects CC_p at point S. If P is the midpoint of BC, prove that points P, S, C_p, C_q lie on the same circle, i.e., the four points are concyclic.



Over my bitter-tainted, trembling lips,

[**5**] *балалайка* has different ranges and pitches due to its different sizes and dimensions. It is comparable to a symphony orchestra when played together in multiple numbers. Even so, all such instruments follow the following rule: the projections of point B on strings AP and AQ are B_p and B_q and, respectively. Connect B_pC_q , B_qC_p and let them intersect at point K. Prove that point K lies on line BC and find out the range of trajectory of point K on line BC.



With melody, deep, clear, and liquid-slow.

[**5**] The resonance chamber of *балалайка* is also a triangle, with points D, E, and F on sides AB, BC, and AC, respectively. Thus, $\triangle DEF$ is the resonance chamber. To achieve the optimal acoustic quality, the incenter of $\triangle ABC$ and the centroid of $\triangle DEF$ need to coincide. Prove that $S_{\triangle DEF}$ reaches its minimum value if and only if the equation $AD+BE+CF=AF+BD+CE$ is satisfied, and express the minimum value of $S_{\triangle DEF}$ as an explicit expression in terms of a, b, c, where $BC=a$, $AC=b$, $AB=c$.

Algebra Part

12. As Furina played better and better, she became increasingly curious about the mathematical principles behind the music.

Tips: Any music theory or physics-related knowledge is neither required nor allowed when solving this question.

The expression of sound waves is $\psi(t) = \sin(2\pi f t)$, where f refers to the frequency.

(**1**) [**8**] The harmonic series consists of a fundamental tone and overtones. If the frequency of the fundamental tone is f, then the frequency of its overtones is $2f$, $3f$, …, nf. Furina wants to investigate the expression for the superimposed sound wave of the harmonic series with a fundamental frequency of $\frac{1}{2\pi}$ Hz, assuming the fundamental tone and its overtones have the same loudness, where n is an integer greater than 1.

① [**3**] Proof:

$$\sum_{k=1}^{n} \sin kx = \frac{\sin\frac{nx}{2}\sin\frac{nx+x}{2}}{\sin\frac{x}{2}}$$

② [**5**] Furina wants to investigate the range of the loudness of the superimposed sound wave of the harmonic series. Let M be the maximum value of $\sum_{k=1}^{n} \sin kx$.

Proof:

$$\frac{2}{3}n < M < n$$

(**2**) [**7**] Furina discovered a continuous function that can roughly compute the loudness $f(x)$ ($f(x) \neq 0$) through the ratio of the maximum vibration distance of a string to its length, x ($-1 \leq x \leq 1$), which satisfies the following equation:

$$f(2x^2 - 1) = \frac{f(x)}{f(x) + 1}$$

Construct one of the functions $f(x)$ that satisfies the above conditions.

Number Theory Part

13. When Number Theory falls in love with Geometry…

Minkowski Theorem is an effective tool connecting number theory and geometry. In the Euclidean space R^n, a bounded centrally symmetric convex body A that satisfies Vol $(A) > 2^n$ must contain a lattice point different from the origin where a lattice point refers to a point where all coordinates are integers and n refers to the dimension. For example, when $n = 2$, the theorem's weaker version states that in the Euclidean plane, any convex closed region that contains the origin and is symmetric about the origin, if its area is greater than 4, must contain a lattice point different from the origin.

（**1**） [**3**] In a three-dimensional space, every lattice point except the origin is the center of a solid carbon nanotube black body sphere with a radius of r. A laser is emitted from the origin, and it will be absorbed when it hits a solid carbon nanotube black body sphere. Prove that any laser will travel a distance of no more than $\frac{e}{2r^2}$ before being absorbed.

（**2**） [**5**] Let $n \in \mathbb{N}^+$. Proof: If the equation $x^2 + xy + y^2 = n$ has rational solutions, the equation has positive integer solutions.

（**3**） [**7**] For each $n \in \mathbb{N}^+$, let $f(n)$ denote the number of methods to express n as a sum of non-negative integer powers of 2 (representations that differ only in the order, such as $2^1 + 2^2$ and $2^2 + 2^1$, are considered the same way. For example, since $4 = 2^2 = 2^1 + 2^1 = 2^1 + 2^0 + 2^0 = 2^0 + 2^0 + 2^0 + 2^0$, we have $f(4) = 4$. Proof that there exist real numbers $a, b, c_1, c_2,$ such that $c_1 n^2 - c_2 n \ln n - an < \ln f(2^n) < c_1 n^2 - c_2 n \ln n - bn$, and solve for c_1, c_2.

Combinatorics Part

14. Math top student Albert and English top student Barbara are not honest. They are going to take an English exam, which consists entirely of multiple-choice questions. They discussed beforehand that Barbara would send signals to Albert with the options during the exam. However, all the questions turned out to be questions with the undetermined number of options. It is known that Albert cannot answer any of the English questions but is very proficient in analysis. Albert knows that Barbara has a habit of writing the options for each question in alphabetical order like A → B → C → D. Assume Barbara can always answer the English questions correctly. For example, suppose the exam has three questions, each with options A, B, and C, and Barbara sent four choice signals to Albert. If the options Barbara sent are "ACBB", Albert can deduce that the answers must be (AC)(B)(B). However, if the options Barbara sent are "ABCC", Albert cannot determine whether the answers are (A)(BC)(C) or (AB)(C)(C). Assume the exam consists of m questions, and Barbara sent n choice signals to Albert, where $m, n \in \mathbb{N}^{+}$.

（**1**） [**7**] Assume that each question has only two options: A and B, with the number of correct options may be either 1 or 2.

① [**1**] Proof: $m \leq n \leq 2m$.

② [**6**] Express, using an explicit expression in terms of m and n, the number of arrangements of options that will definitely enable Albert to achieve full marks in the exam.

（**2**） [**8**] Assume that each question has four options: A, B, C, and D, with the number of correct options possibly being 1, 2, 3, or 4. Express, using an explicit expression in terms of m and n, the number of arrangements of options that will definitely enable Albert to achieve full marks in the exam.